

# Machine Learning Theory

## Least Squares in Infinite Models i.e. Regression

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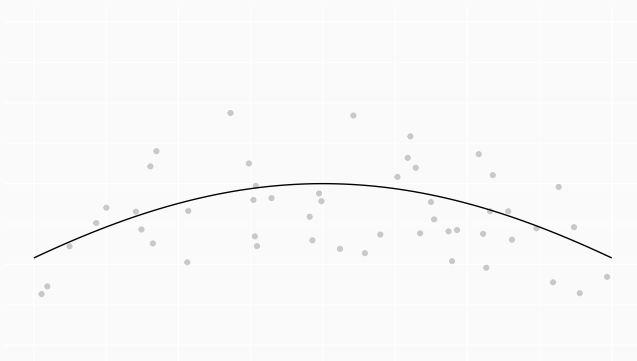
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# Least squares with gaussian noise

We observe  $Y_i = \mu(X_i) + \epsilon_i$  for  $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$ .



We're estimating the curve  $\mu(x)$ .  
Our goal is get close in terms of *sample mean squared distance*.

This is the kind of statement we're after.

$$\|\hat{\mu} - \mu\|_{L_2(P_n)} < s \quad \text{with probability} \quad 1 - \delta$$

## Old Friends

- $(X_i, Y_i)$  for  $i = 1 \dots n$ . The data.
- $\mu(x)$ , the estimation target. A curve.
- $\mathcal{M}$ , the model. A set of curves.  
For today, a *convex set* containing infinitely many curves.
- $\hat{\mu}$ , our estimate. Some curve in the model, chosen because it fits the data.
- $m$ , an anonymous curve. Whatever curve we're thinking about at the moment.

## New Ones

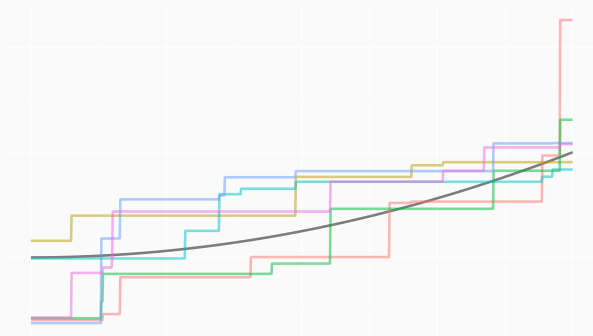
- $\mathcal{M}_s$ , a *neighborhood* of the target.
  - It's the subset of curves in our model that are close to  $\mu$ .
  - We're trying to show that  $\hat{\mu}$  is in it.
- $\mathcal{M} \setminus \mathcal{M}_s$ , its complement.
  - It's the subset of curves in our model that aren't close to  $\mu$ .
  - It's equivalent to show that  $\hat{\mu}$  is *not* one of the curves in it.
- $\mathcal{M}_s^\circ$ , the boundary of the neighborhood  $\mathcal{M}_s$ .
  - This will play a special role in *convex models*.
  - That's what we'll be talking about today.

For now, we'll think of  $X_1 \dots X_n$  as deterministic.

If they are random, we *condition* on them.

## What's changed from last week is our model

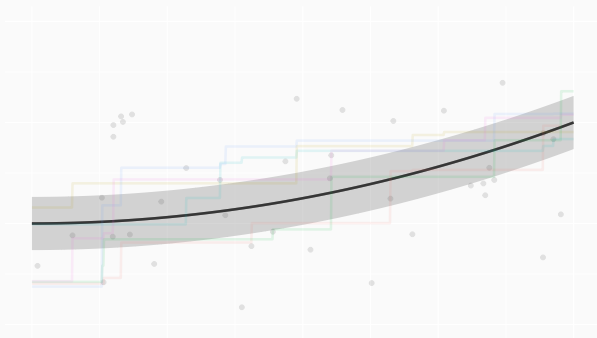
Last week, our model was a finite set of curves.



Like these.

# What's changed from last week is our model

Last week, our model was a finite set of curves.



A neighborhood is the subset of these curves that's close enough to  $\mu$ .  
Say within the gray tube.

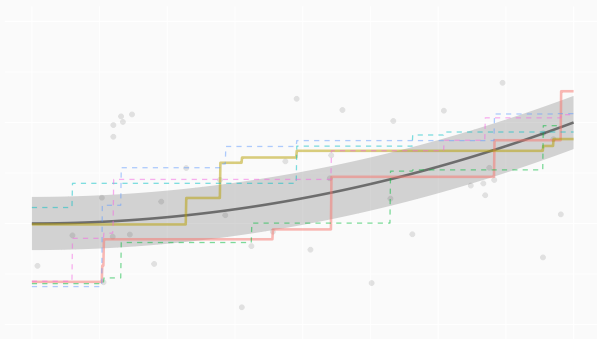
**Caveat.**

The gray tube is the set of curves that are close in terms of the infinity norm.

$$\mathcal{M}_s^\infty = \{m \in \mathcal{M} : \|m - \mu\|_\infty < s\}$$

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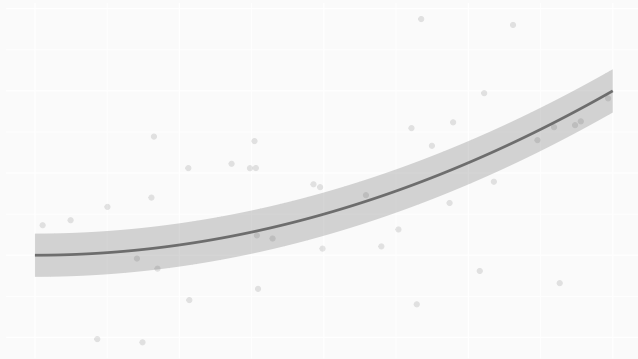
We're talking about the set of curves that are close in terms of the sample two-norm.

$$\mathcal{M}_s = \{m \in \mathcal{M} : \|m - \mu\|_{L_2(\mathbb{P}_n)} < s\}$$

Think of these as curves that are mostly, but not necessarily always, in the tube.  
These are plotted as solid lines above. Those in the complement are dashed.

# Neighborhoods in models with infinitely-many curves

Let's take the set of increasing curves to be our regression model.



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Say within the gray tube.

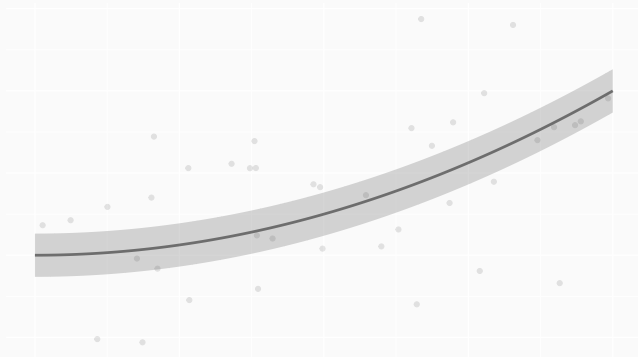
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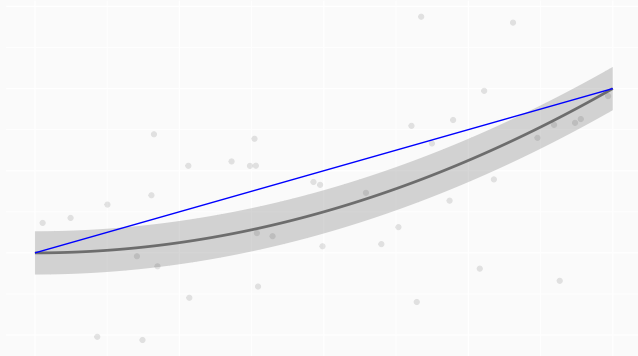
Now that our model has infinitely many curves, we can't draw all of them.

Let's look at a few examples instead.



# Neighborhoods in models with infinitely-many curves

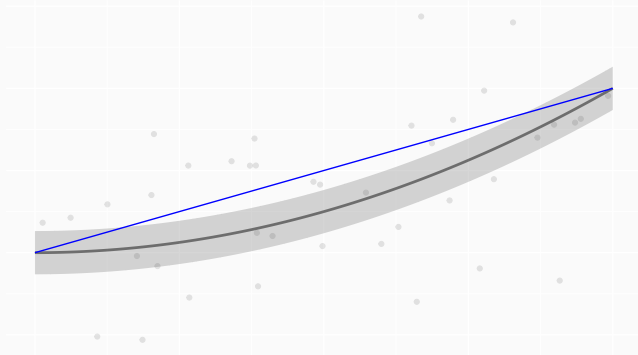
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Is this in our neighborhood?

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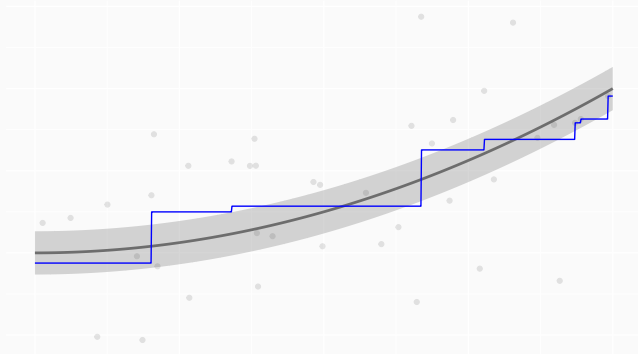


Is this in our neighborhood?

No. It's too far from  $\mu$

# Neighborhoods in models with infinitely-many curves

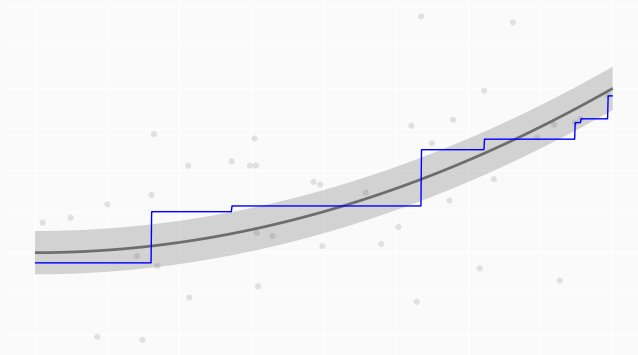
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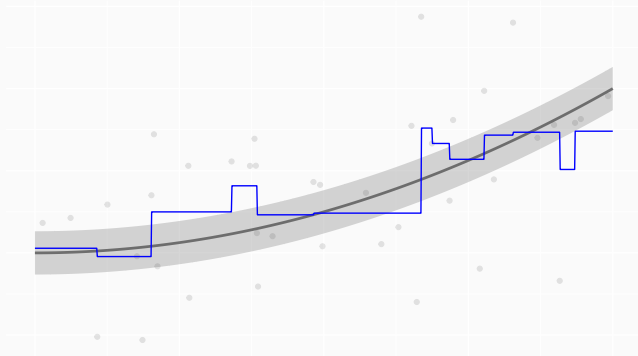


Is this in our neighborhood?

Yes.

# Neighborhoods in models with infinitely-many curves

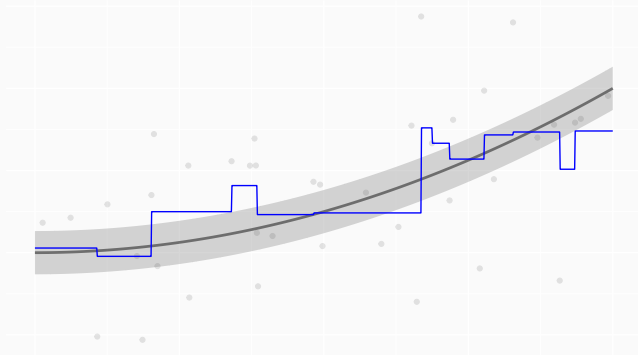
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Is this in our neighborhood?

# Neighborhoods in models with infinitely-many curves

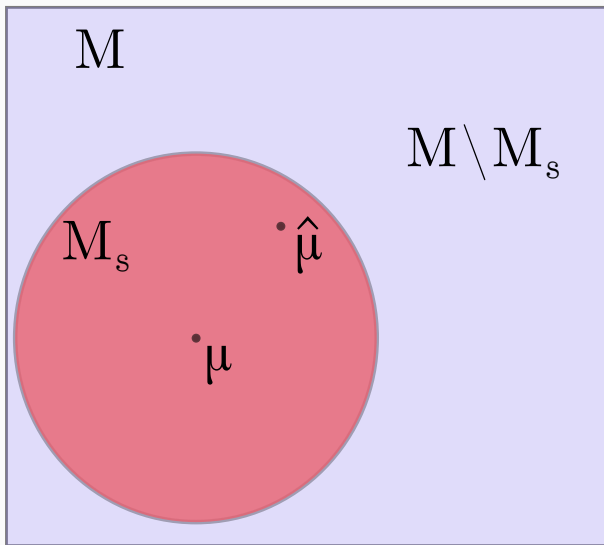
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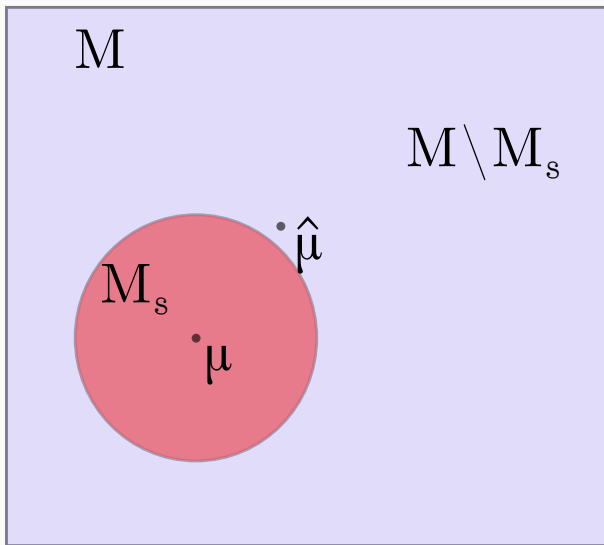
Is this in our neighborhood?

No. It's close to  $\mu$ , but it's not in our model. It's not increasing.

# What we're going to prove today



## What we're going to rule out





## The argument in words

What we know is that  $\hat{\mu}$  beats or ties every other curve in the model.  
That's what a minimizer (argmin) does.

$$\hat{\mu} = \operatorname{argmin}_{m \in \mathcal{M}} \ell(m) \iff \ell(\hat{\mu}) \leq \ell(m) \text{ for all } m \in \mathcal{M}$$

If **our model is right**, that means it beats or ties  $\mu$ .

$$\ell(\hat{\mu}) \leq \ell(m) \text{ for all } m \in \mathcal{M} \text{ and } \mu \in \mathcal{M} \implies \ell(\hat{\mu}) \leq \ell(\mu).$$

And if **no curve in our neighborhood's complement beats or ties  $\mu$** ,  
this means  $\hat{\mu}$  isn't in that complement.

$$\ell(\hat{\mu}) \leq \ell(\mu) \text{ and } \ell(m) > \ell(\mu) \text{ for all } m \in \mathcal{M} \setminus \mathcal{M}_s \implies \hat{\mu} \notin \mathcal{M} \setminus \mathcal{M}_s$$

And because  $\hat{\mu}$  is in the model, that means  $\hat{\mu}$  is in the neighborhood.

$$\hat{\mu} \notin \mathcal{M} \setminus \mathcal{M}_s \text{ and } \hat{\mu} \in \mathcal{M} \iff \hat{\mu} \in \mathcal{M}_s$$

When our **two if clauses** are true, this argument implies  $\hat{\mu}$  is in our neighborhood.  
So if they're true with some probability,  $\hat{\mu}$  is in the neighborhood with that probability.

Today we'll assume we got the model right, so the **second if** is what we need to prove.

## A Reduction

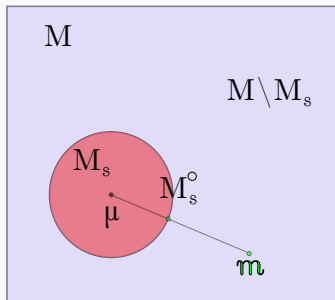
Simplifying our proof for convex models.

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# What Convexity Buys Us

- When the model is a *convex set*, we needn't worry about most of the complement.
- If there's no curve on the boundary with squared loss less than  $\mu$ 's, there's none in the rest of the complement, either.

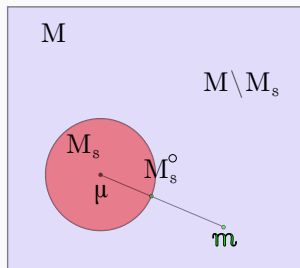
$$\begin{array}{ccc} \ell(m) > \ell(\mu) & \text{for all } m \in \mathcal{M}_s^\circ & \implies \ell(m) > \ell(\mu) \text{ for all } m \in \mathcal{M} \setminus \mathcal{M}_s. \\ \text{the thing we're going to prove} & & \text{the thing we said we needed to prove} \end{array}$$



- Think of a curve in the complement as having a representative on the boundary.
- To find it, draw a line from the curve toward  $\mu$ . Stop where you hit the boundary.
- A curve's loss is *always* bigger than  $\mu$ 's if its representative's is.
- So if the representative of every curve in the complement has loss bigger than  $\mu$ 's, so does every curve in the complement.

# Representatives do it for their constituents

**Proof.** A curve's squared error loss is *always* bigger than  $\mu$ 's if its representative's is.



A representative is a point  
 $m_t = \mu + t(m - \mu)$  for some  $t \in [0, 1]$ .

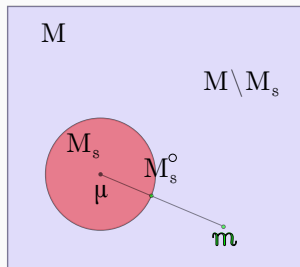
We'll show the loss difference  $\ell(m) - \ell(\mu)$  for any curve in the complement is at least  $t$  **times** the loss difference  $\ell(m_t) - \ell(\mu)$  for its representative.

$$\ell(m_t) - \ell(\mu) \leq t\{\ell(m) - \ell(\mu)\}$$

This means that if the representative's is positive, so is the original curve's.

# Representatives do it for their constituents

**Proof.** A curve's squared error loss is *always* bigger than  $\mu$ 's if its representative's is.

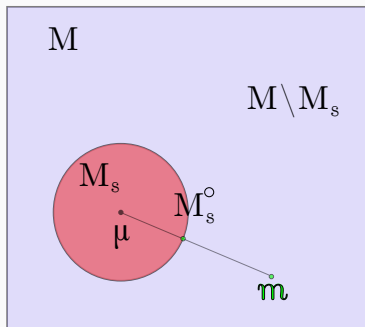


A representative is a point  
 $m_t = \mu + t(m - \mu)$  for some  $t \in [0, 1]$ .

$$\begin{aligned}\ell(m_t) - \ell(\mu) &= \|m_t - \mu\|_{L_2(P_n)}^2 - 2\langle \varepsilon, m_t - \mu \rangle_{L_2(P_n)} \\ &= \|\mu + t(m - \mu) - \mu\|_{L_2(P_n)}^2 - 2\langle \varepsilon, \mu + t(m - \mu) - \mu \rangle_{L_2(P_n)} \\ &= t^2 \|m - \mu\|_{L_2(P_n)}^2 - 2t \langle \varepsilon, m - \mu \rangle_{L_2(P_n)} \\ &\leq t \{ \ell(m) - \ell(\mu) \} \quad \text{because} \quad t^2 \leq t.\end{aligned}$$

# What is Convexity?

Convexity is a property of a model that guarantees that, no matter what  $\mu \in \mathcal{M}$  is, each curve in a neighborhood's complement has a representative on its boundary.

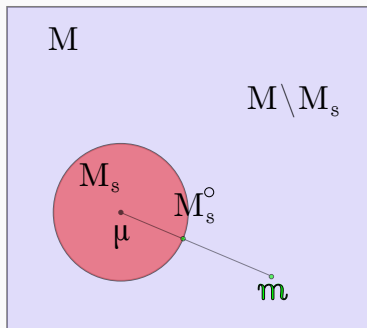


## Definition.

A set is convex **if and only if** it contains the line between any two of its points.

# Why do we need Convexity?

Why isn't ruling out the boundary necessarily enough to rule out the complement if the model isn't convex?

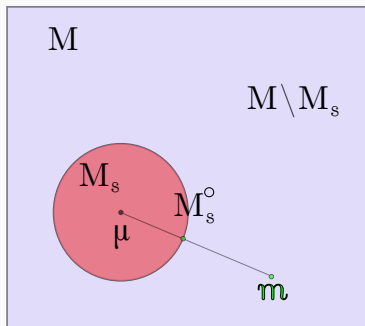


Hint.

Convexity is a property of a model that guarantees that, no matter what  $\mu \in \mathcal{M}$  is, each curve in a neighborhood's complement has a representative on its boundary.

# Why do we need Convexity?

Why isn't ruling out the boundary necessarily enough to rule out the complement if the model isn't convex?



In a nonconvex model, there may be curves in the complement without representatives on the boundary. Ruling out the boundary doesn't cover them.



## Following through.

Proving the simplified claim.

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Throughout,  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  will be the  $L_2(\mathbf{P}_n)$  norm and inner product.

$$\|f\|^2 = \frac{1}{n} \sum_{i=1}^n f(X_i)^2$$

$$\langle f, g \rangle = \frac{1}{n} \sum_{i=1}^n f(X_i)g(X_i).$$

# The Deterministic Part

What we're proving is a *lower bound* on differences in mean squared error.

$$\ell(m) > \ell(\mu) \quad \text{or equivalently} \quad \ell(m) - \ell(\mu) > 0 \quad \text{for all} \quad m \in \mathcal{M}_s^\circ.$$

And we only need to bother with curves on the neighborhood's boundary.

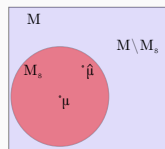
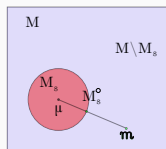
$$\ell(m) - \ell(\mu) = \underbrace{\|m - \mu\|^2}_{=s^2} - 2\langle \varepsilon, m - \mu \rangle \quad \text{for} \quad m \in \mathcal{M}_s^\circ$$

Nice! All we have to do is bound the mean zero term for all curves on the boundary.

$$\begin{aligned} \text{The difference is positive if} \quad & s^2/2 > \langle \varepsilon, m - \mu \rangle \quad \text{for all} \quad m \in \mathcal{M}_s^\circ \\ \text{or equivalently if} \quad & s^2/2 > \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle. \end{aligned}$$

This implies that every curve in the complement  $\mathcal{M} \setminus \mathcal{M}_s$  has bigger loss than  $\mu$ .

When it's satisfied, we know that  $\hat{\mu} \in \mathcal{M}_s$ , i.e. that  $\|\hat{\mu} - \mu\| < s$ .



# Taking Advantage of Approximate Constancy

- Our error bound holds with high probability if our  $s^2/2 \geq \max \dots$  inequality does.

$$\|\hat{\mu} - \mu\| < s \text{ when } \frac{s^2}{2} > \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle \quad \text{so}$$

$$P(\|\hat{\mu} - \mu\| < s) \geq P\left(\frac{s^2}{2} > \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle\right)$$

- We want to choose  $s$  so this happens with probability (at least)  $1 - \delta$ .
- It helps that this maximum is *approximately constant*. For  $M_n = 1 + 2 \log(2n)$ ,

$$\left| \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle - \mathbb{E} \left[ \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle \right] \right| \leq s\sigma \sqrt{\frac{2M_n}{\delta n}} \quad \text{w.p. } 1 - \delta.$$

- It's almost always close to its expected value. And the way it differs is simple.
  - We can bound the difference without thinking about the model  $\mathcal{M}$ .
  - And our bound is small unless  $\delta$  is very small, i.e. unless we want too much certainty.
- We'll use this to *sandwich* a bound between  $s^2/2$  and  $\max \dots$  above.

$$\frac{s^2}{2} \stackrel{(a)}{\geq} \mathbb{E} \left[ \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle \right] + s\sigma \sqrt{\frac{2M_n}{\delta n}} \stackrel{(b)}{\geq} \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle \quad \text{w.p. } 1 - \delta.$$

- (a). We choose  $s$  so this inequality is satisfied.
  - We have to do this every time we consider a new model.
  - But we don't have to worry about randomness. Both sides are deterministic.
- (b). This inequality follows from our 'approximate constancy' result.
  - We'll only have to prove that once. It's true for every model.
  - We'll do that next class using, and proving, the Efron-Stein inequality.

## Intuition on Approximate Constancy

Why is  $\left| \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle - \mathbb{E} \left[ \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle \right] \right| \leq s\sigma \sqrt{\frac{2M_n}{\delta n}}$  w.p.  $1-\delta$  ?

- The difference won't be too big compared to the maximum's standard deviation.

$$|Z - \mathbb{E} Z| < \frac{\text{sd}(Z)}{\sqrt{\delta}} \quad \text{w.p. } 1-\delta \quad \text{for any random variable } Z. \quad [\text{Chebyshev's Inequality}]$$

- So what's that standard deviation? Or what's the variance?
  - It's the variance of a maximum of sample means: one for each function  $m$  in  $\mathcal{M}_s^\circ$ .
  - So, as a starting point, let's bound the variance of a single one.

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  - It's the variance of a maximum of sample means: one for each function  $m$  in  $\mathcal{M}_s^\circ$ .
  - So, as a starting point, let's bound the variance of a single one.

$$\begin{aligned} \text{Var} \left[ \frac{1}{n} \sum_{i=1}^n \varepsilon_i \{m(X_i) - \mu(X_i)\} \right] &= \frac{1}{n^2} \sum_{i=1}^n \mathbb{E} \varepsilon_i^2 \{m(X_i) - \mu(X_i)\}^2 \\ &= \frac{\sigma^2}{n} \times \|m - \mu\|^2 \\ &= \frac{\sigma^2}{n} \times s^2 \quad \text{for } m \in \mathcal{M}_s^\circ. \end{aligned}$$

- Next class, we'll show that the variance of the maximum isn't much larger.

$$\text{Var} \left[ \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle \right] \leq 2 \frac{M_n}{1+2 \log(2n)} \times \frac{\sigma^2}{n} \times s^2 \quad \text{so} \quad \text{sd}(\dots) \leq s\sigma \sqrt{\frac{2M_n}{n}}$$

- Plugging it into Chebyshev's Inequality, we get our approximate constancy result.

## Summary

$$\|\hat{\mu} - \mu\| < s \quad \text{w.p.} \quad 1 - \delta \quad \text{if} \quad \frac{s^2}{2} > \mathbb{E} \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon, m - \mu \rangle + s\sigma \sqrt{\frac{2M_n}{\delta n}}.$$

Let's introduce some notation to say this a bit more compactly.

Here's our characterization of  $s$  rephrased in terms of  $g_i \stackrel{iid}{\sim} N(0, 1)$ .

$$\frac{s^2}{2\sigma} \geq w(\mathcal{M}_s^\circ - \mu) + s\sqrt{\frac{2M_n}{\delta n}} \quad \text{where}$$

$\mathcal{M}_s^\circ - \mu := \{m - \mu : m \in \mathcal{M}_s^\circ\}$  is the *centered neighborhood boundary*,

$w(\mathcal{V}) := \mathbb{E} \max_{v \in \mathcal{V}} \langle g, v \rangle$  is the Gaussian width of the set  $\mathcal{V}$ .

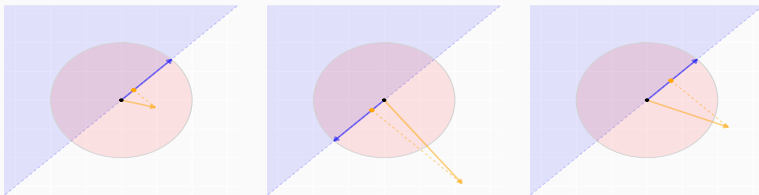
The *Gaussian width*  $w(\mathcal{V})$  of a set of vectors  $\mathcal{V} \subseteq \mathbb{R}^n$  is the expectation of the largest sample inner product between

- any vector  $v \in \mathcal{V}$
- a vector  $g$  of independent standard normals

As a shorthand, we'll talk about the Gaussian width of a set of functions.

We'll mean the set of vectors we get when we evaluate them at  $X_1 \dots X_n$ .

# Visualizing Gaussian Width



- The gaussian width  $w(\mathcal{M}_s^\circ)$  is the expected value of something.
  - The maximum inner product between a gaussian vector  $g$  and a vector  $v \in \mathcal{M}_s^\circ$ .
  - We can think of this as the average of this maximum when we sample  $g$  over and over.

$$w(\mathcal{M}_s^\circ) = \mathbb{E} \max_{v \in \mathcal{M}_s^\circ} \langle g, v \rangle \approx \frac{1}{m} \sum_{j=1}^m \max_{v \in \mathcal{M}_s^\circ} \langle g^{(j)}, v \rangle \quad \text{for } g_i^{(j)} \stackrel{iid}{\sim} N(0, 1).$$

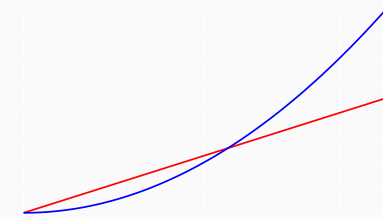
- That's something we can draw and look at. I've drawn it three times above.
- In each,  $\mathcal{M}$  is the blue region and  $\mathcal{M}_s^\circ - \mu$  is the rim of the red semicircle. And ...
  - I've sampled a gaussian vector  $g$ .
  - Then found the vector  $v$  in  $\mathcal{M}_s^\circ$  maximizing the inner product  $\langle g, v \rangle$ .
  - It's  $\|v\| = s$  times the length  $\langle g, v/\|v\| \rangle$  of the projection indicated by the orange dot.
- The width of  $\mathcal{M}_s^\circ$  is the average inner product  $\langle g, v \rangle$  in infinitely many plots like these.



## A Simplification

$$\|\hat{\mu} - \mu\| < s \quad \text{w.p.} \quad 1 - \delta \quad \text{if} \quad \frac{s^2}{2\sigma} > w(\mathcal{M}_s^\circ - \mu) + s\sqrt{\frac{2M_n}{\delta n}}.$$

The right side's second term tends to be small. We can ignore it and be vague.



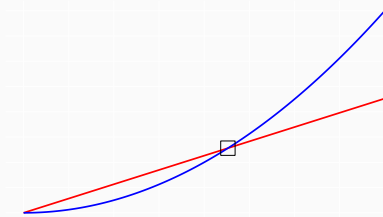
$\|\hat{\mu} - \mu\|$  isn't much bigger than  $s$  with high probability if  $s^2 \geq 2\sigma w(\mathcal{M}_s^\circ - \mu)$ .

What is the smallest  $s$  for which this is true?

## A Simplification

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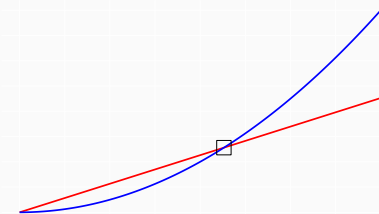
What is the smallest  $s$  for which this is true?

The point at which the red and blue curves intersect.

# A Simplification

$$\|\hat{\mu} - \mu\| < s \quad \text{w.p.} \quad 1 - \delta \quad \text{if} \quad \frac{s^2}{2\sigma} > w(\mathcal{M}_s^\circ - \mu) + s\sqrt{\frac{2M_n}{\delta n}}.$$

The right side's second term tends to be small. We can ignore it and be vague.

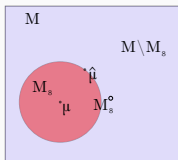


But we can boil things down to the same picture and still be precise.

$$\|\hat{\mu} - \mu\| < s + 2\sigma\sqrt{\frac{2M_n}{\delta n}} \quad \text{w.p.} \quad 1 - \delta \quad \text{if} \quad s^2 \geq 2\sigma w(\mathcal{M}_s - \mu)$$

- We'll prove it for homework. It's not hard to derive from the first bound.
- The key idea is that  $w(\mathcal{M}_s - \mu)$  is a *sublinear* function of  $s$ .

# Caveat



$$\|\hat{\mu} - \mu\| < s$$

$$\text{w.p. } 1 - \delta \text{ if } \frac{s^2}{2\sigma} \geq \mathbf{w}(\mathcal{M}_s^\circ - \mu) + s\sqrt{\frac{2M_n}{\delta n}}$$

$$\|\hat{\mu} - \mu\| < s + 2\sigma\sqrt{\frac{2M_n}{\delta n}}$$

$$\text{w.p. } 1 - \delta \text{ if } s^2 \geq 2\sigma \mathbf{w}(\mathcal{M}_s - \mu)$$

- In our second bound, we use the width of the centered neighborhood  $\mathcal{M}_s - \mu$  instead of its boundary  $\mathcal{M}_s^\circ - \mu$ .
  - We need to do this because the second requires width to grow sublinearly with  $s$ .
  - When  $\mathcal{M}_s$  is convex, this is true for  $\mathbf{w}(\mathcal{M}_s - \mu)$ , but not necessarily for  $\mathbf{w}(\mathcal{M}_s^\circ - \mu)$ .
- This width is larger because it's a maximum over a larger (i.e. containing) set.

$$\mathbb{E} \max_{m \in \mathcal{M}_s} \langle g, m - \mu \rangle \geq \mathbb{E} \max_{m \in \mathcal{M}_s^\circ} \langle g, m - \mu \rangle \quad \text{because } \mathcal{M}_s \supseteq \mathcal{M}_s^\circ.$$

$\mathbf{w}(\mathcal{M}_s - \mu)$ 
 $\mathbf{w}(\mathcal{M}_s^\circ - \mu)$

- But usually we don't pay much—if anything—for making this substitution.
- When we try to bound  $\mathbf{w}(\mathcal{M}_s - \mu)$ , we'll often get one that applies to  $\mathbf{w}(\mathcal{M}_s)$  too.
- If it does matter, and we really want to work with  $\mathbf{w}(\mathcal{M}_s^\circ - \mu)$ , we can.
  - We'll just need to use the first (messier)  $s^2 \geq \max \dots$  bound.
  - Or take a different approach to simplifying it that doesn't require sublinearity.

## Implications

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## Example 1

Suppose we have a width bound that is proportional to  $s$ :  $\alpha s \geq w(\mathcal{M}_s - \mu)$ .

$$\begin{aligned} s^2 &\geq 2\sigma w(\mathcal{M}_s - \mu) \text{ if } s^2 \geq 2\sigma\alpha s \\ &\text{and therefore if } s \geq 2\sigma\alpha. \end{aligned}$$

This is what happens when we use a model with  $K$  parameters.

$$c\sqrt{\frac{K}{n}} s \geq w(\mathcal{M}_s - \mu) \quad \text{and therefore} \quad s = 2\sigma c\sqrt{\frac{K}{n}} \quad \text{works.}$$

- One interesting thing we can do with this is see what happens when we choose model size as a function of sample size.
- Our estimator converges when our model's size is much smaller than sample size.

$$\begin{aligned} \|\hat{\mu} - \mu\| &\leq 2\sigma c\sqrt{\frac{K}{n}} + 2\sigma\sqrt{\frac{2M_n}{\delta n}} \quad \text{w.p. } 1 - \delta \\ &\rightarrow 0 \quad \text{if } K/n \rightarrow 0 \end{aligned}$$

This and finite-dimensional approximation get you pretty far with infinite-dimensional models, too.

## Example 2

Suppose we have a width bound that doesn't depend on  $s$ :  $\beta \geq w(\mathcal{M}_s - \mu)$ .

$$\begin{aligned} s^2 \geq 2\sigma w(\mathcal{M}_s - \mu) \text{ if } s^2 \geq 2\sigma\beta \\ \text{and therefore if } s \geq \sqrt{2\sigma\beta}. \end{aligned}$$

This is what happens when our model is the set of *weighted averages* of  $K$  functions.

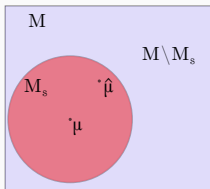
$$w(\mathcal{M}_s - \mu) \leq \sqrt{c \log(K)/n} \quad \text{and therefore} \quad s = \sqrt{2} \sqrt[4]{c\sigma \log(K)/n} \quad \text{works.}$$

- We get a  $n^{-1/4}$  rate essentially independent of the number of functions  $K$ .
- This is at the heart of the Assumptionless Analysis of the Lasso.

$$\begin{aligned} \|\hat{\mu} - \mu\| &\leq \sqrt{2} \sqrt[4]{c\sigma \log(K)/n} + 2\sigma \sqrt{\frac{2M_n}{\delta n}} \quad \text{w.p.} \quad 1 - \delta \\ &\rightarrow 0 \quad \text{if} \quad \log(K)/n \rightarrow 0. \end{aligned}$$

With high probability ...

the error of the least squares estimator in a convex model  $\mathcal{M}$  is smaller than a radius  $s$  determined by the Gaussian width of the *centered neighborhood boundary*.



$$\|\hat{\mu} - \mu\|_{L_2(P_n)} \leq s + 2\sigma \sqrt{\frac{2M_n}{\delta n}}$$

with probability  $1 - \delta$  if

$$s^2 \geq 2\sigma w(\mathcal{M}_s - \mu)$$

- That's true if  $\mu$  is in the model.
  - If it's not, it's true about distance to the best approximation to  $\mu$  in the model.
  - We'll talk about that next lecture.
- It's also true that this is about as good a guarantee as you can get.
- This means that the study of least squares estimation is, for the most part, just the study of Gaussian width.



## Gaussian Width

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## Gaussian Width

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An Example

The Completely General Model

# The General Model

Consider a model  $\mathcal{M}$  that contains every curve outright.

$$\mathcal{M} = \{ \text{all curves } m(x) \}$$

Or, as we're interested in its values on the sample, the corresponding set of vectors.

$$\mathcal{M} = \{ \text{all vectors } \vec{m} \in \mathbb{R}^n : \vec{m}_i = \vec{m}_j \text{ if } X_i = X_j \} \subseteq \{ \text{all vectors } \vec{m} \in \mathbb{R}^n \} = \tilde{\mathcal{M}}.$$

To keep things simple, we can enlarge this set by dropping the **function constraint**.

A neighborhood is just a ball centered on  $\vec{\mu} = \mu(X_1) \dots \mu(X_n)$ .

$$\tilde{\mathcal{M}}_s = \{ \vec{m} : \|\vec{m} - \vec{\mu}\|_{L_2(\mathbf{P}_n)} \leq s \}$$

And when we center it, we get a ball around the origin.

$$\tilde{\mathcal{M}}_s - \mu = \{ \vec{m} - \vec{\mu} : \|\vec{m} - \vec{\mu}\|_{L_2(\mathbf{P}_n)} \leq s \} = \{ v \in \mathbb{R}^n : \|v\|_{L_2(\mathbf{P}_n)} \leq s \}.$$

# The General Model's Gaussian Width

$$w(\tilde{\mathcal{M}}_s - \mu) = \mathbb{E} \max_{\substack{v \in \mathbb{R}^n \\ \|v\|_{L_2(\mathbf{P}_n)} \leq s}} \langle g, v \rangle_{L_2(\mathbf{P}_n)}$$

No matter what gaussian vector we get, this set has a vector  $v$  with 2-norm  $s$  in exactly the same direction. It's got one in every direction.

$$w(\tilde{\mathcal{M}}_s - \mu) = \mathbb{E} \max_{v \in \tilde{\mathcal{M}}_s} \|g\|_{L_2(\mathbf{P}_n)} \|v\|_{L_2(\mathbf{P}_n)} = s \mathbb{E} \|g\|_{L_2(\mathbf{P}_n)}.$$

And the expected sample two-norm of a gaussian vector  $g \in \mathbb{R}^n$  is roughly 1. That's the square root of its expected *squared* sample two-norm.

$$\mathbb{E} \|g\|_{L_2(\mathbf{P}_n)} \approx \sqrt{\mathbb{E} \|g\|_{L_2(\mathbf{P}_n)}^2} = \sqrt{\frac{1}{n} \sum_{i=1}^n \mathbb{E}[g_i^2]} = \sqrt{1}.$$

So the gaussian width we get is roughly just  $s$ . Let's call it  $s$ . Our bound doesn't tell us that we'll ever get close to  $\mu$  at all.

$$s^2 \geq 2\sigma w(\mathcal{M}_s - \mu) \quad \text{if} \quad s^2 \geq 2\sigma s \quad \text{i.e. if} \quad s \geq 2\sigma.$$

Wouldn't that be too much to expect—to get close without any assumptions at all?

## Lab Preview

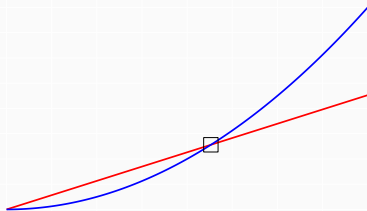
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# Computing Gaussian Width

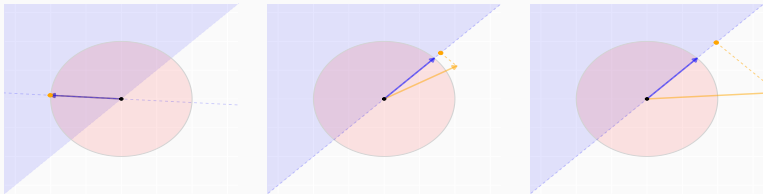
- Gaussian width is the mean of something we can compute samples of.
- That means we can approximate it by a sample average.

$$w(\mathcal{V}) = \mathbb{E} \max_{v \in \mathcal{V}} \langle g, v \rangle \approx \frac{1}{m} \sum_{j=1}^m \max_{v \in \mathcal{V}} \langle g^{(j)}, v \rangle \quad \text{for } g_i^{(j)} \stackrel{iid}{\sim} N(0, 1).$$

- This means we can calculate a neighborhood's Gaussian width whether we can work out how to do it analytically or not.
  - Next week, we'll use CVXR to do it.
  - And search over the radius  $s$  to find an error bound.



$$\|\hat{\mu} - \mu\| < s + 2\sigma \sqrt{\frac{2M_n}{\delta n}} \quad \text{w.p.} \quad 1 - \delta \quad \text{if} \quad s^2 \geq 2\sigma w(\mathcal{M}_s - \mu)$$



- To prepare, we'll do a drawing exercise to get a feel for gaussian width.
  - We'll work out what vectors are in some models when we have just two observations.
  - And use our drawings, as described earlier, to calculate width in the 2 observation case.
- We'll vary the radius  $s$  to search for an error bound, too.